Critical Evaluation of Equivalent Moment Factor Procedures for Laterally Unsupported Beams

EDGAR WONG and ROBERT G. DRIVER

ABSTRACT

This paper compares the numerous approaches to determining equivalent moment factors used in evaluating the elastic critical moment of laterally unsupported beams for a wide variety of moment distributions. The investigation revealed that the procedure used currently in the Canadian design standard produces unacceptable results for the majority of the common bending moment distributions considered. Large abrupt changes in C_b values with only slight changes in the shape of the moment diagram were observed in 6 out of the 12 moment distribution comparisons, which contributes to the overall poor performance of the procedure.

The study also revealed drawbacks inherent in other methods. Overall, the quarter-point moment equations developed for general moment distributions capture the trends of the numerical data reasonably well. However, for example, the evaluations show that the 2005 AISC equation produces non-conservative results in some situations, while the British equation, although generally conservative, produces comparatively less accurate results. Other equations examined capture the trends of the numerical data more consistently by implementing a square root format in the quarter-point moment method. However, they produce results that exceed the numerical data in several cases, implying that they are too aggressive for design purposes.

To capture the best features of the various methods investigated, yet improve the overall suitability for general design purposes, a modified quarter-point moment equation using the square root format is proposed. Not only does it simulate the trends of the numerical solutions closely, but it also produces reasonable and conservative equivalent moment factors, even in cases where other methods do not. Like all quarter-point moment methods, the proposed equation does not produce good results in some situations where concentrated moments are applied. Nevertheless, it is believed to be appropriate for the vast majority of typical design cases.

Keywords: lateral support, equivalent moment factors, C_b, beams.

INTRODUCTION

The elastic lateral-torsional buckling moment capacity of a doubly-symmetric steel beam is governed primarily by the member's weak-axis moment of inertia, I_y, and the torsion parameter. The latter factor can be expressed as

$$\frac{\pi}{L}\sqrt{\frac{EC_w}{GJ}}$$

where L is the laterally (and torsionally) unbraced length of the beam, E is the elastic modulus, G is the shear modulus, C_w is the warping constant, and J is the torsional constant. Previous research has shown that the following factors can also influence the critical moment capacity significantly (Clark and Hill, 1960; Nethercot and Rockey, 1972; Nethercot and Trahair, 1976):

1. The internal moment distribution between brace points;
2. The elevation of the applied load with respect to the shear center;
3. The degree of lateral, rotational, and warping restraint at the brace points; and
4. The potential for less critical adjacent unbraced segments to restrain buckling (i.e., interaction buckling).

Although methods that consider all of these factors in the computation of the elastic critical moment are available, most steel design specifications simplify the analytical process by accounting only for the moment distribution effect among the four factors. That is, loads are assumed to be applied at the shear center (unless, perhaps, they are applied significantly above the shear center (for downward loads) by a means that does not also serve as a brace), lateral braces are assumed to prevent both lateral displacement and twist of the beam's cross section, while restraining neither weak-axis rotation nor warping, and the potential for interaction buckling is neglected. For these reasons, this paper addresses the effect of the moment distribution only.

Edgar Wong, M.Eng., P.Eng., structural engineer, Walters Chambers and Associates Ltd., 501, 10709 Jasper Ave., Edmonton, AB, T5J 3N3, Canada. E-mail: edgar.wong@walterschambers.com

Robert G. Driver, Ph.D., P.Eng., Professor, Department of Civil and Environmental Engineering, University of Alberta, Edmonton, AB, T6G 2W2, Canada (corresponding author). E-mail: rdriver@ualberta.ca

For a doubly-symmetric beam subject to a uniform (constant) moment about the strong axis along its length, the critical lateral-torsional buckling capacity, M_{cr}, can be expressed as:

$$M_{cr} = \frac{\pi}{L}\sqrt{EI_y GJ + \left(\frac{\pi E}{L}\right)^2 I_y C_w} \tag{1}$$

The boundary conditions assumed in this equation are such that both ends of the unbraced segment are restrained as described in the previous paragraph. The value determined from Equation 1 is commonly referred to as the "basic" lateral-torsional buckling moment capacity, not only because it is the simplest to derive theoretically, but, more importantly, it gives the lowest possible capacity of a beam segment between properly designed brace points when loads are applied at the shear center (Kirby and Nethercot, 1979). It is widely accepted that the effect of a non-uniform moment distribution can be approximated simply by multiplying Equation 1 by an equivalent (uniform) moment factor, C_b. Since a non-uniform moment distribution is less severe than a uniform one, the value of this factor is always greater than or equal to 1.0.

In general, non-uniform moment distributions between brace points can conveniently be categorized into three groups: (1) linear moment distributions arising when there are no loads or moments applied between brace points; (2) non-linear moment distributions with multiple constant moment gradient regions; and (3) non-linear moment distributions with continuously varying moment gradients. The primary difference between the last two groups is that beams within group 2 are not subjected to any distributed load and their moment distributions can be transformed into group 1 distributions by adding braces at points where the moment gradient changes. It is important to realize that some existing equivalent moment factor equations have been derived for group 1 moment distributions only, whereas others purport to be applicable for all groups. Misusing the equations may lead to significant errors in critical moment predictions.

It should be noted for clarity that in many cases the means of delivering loads to a beam will also provide effective lateral bracing to that beam, apparently making group 1 moment distributions the only case that will occur in practice. However, circumstances where loads are applied to a beam with little, or perhaps uncertain, resulting bracing effectiveness are relatively common. One example of this is where two parallel primary beams have loads delivered to them by simply supported transverse secondary framing members (i.e., the two primary beams "lean on" each other with respect to the intermediate lateral support points at the ends of the secondary members). If the two primary beams become unstable at a similar time in the loading regime, they cannot be considered to support each other laterally (Galambos, 1998). Another common example is where a secondary member delivers its reaction load to the primary beam away from its compression flange without providing significant rotational restraint to the beam and the designer deems this to be inadequate as a bracing mechanism. Other typical examples of loads that are not associated with the provision of effective bracing include suspended loads, supported column reaction loads and loads where the connection of the tributary beam to the primary beam is bolted and employs horizontally slotted holes. Considering the multitude of conditions that a structural designer may face, group 1, 2 and 3 moment distributions must all be included in any evaluation of equivalent moment factors.

There are many equations and methods published in the general literature and design specifications for determining equivalent moment factors. In this paper, comparisons of equivalent moment factors determined using various methods for 12 different moment distribution types, described in Table 1, are presented. In order to generalize the moment distribution types, three factors are introduced in Table 1: (1) for Type 1, the factor κ is the ratio of the absolute value of the smaller to larger end moment of the unbraced segment, and it is taken as positive for double curvature bending and negative for single curvature; (2) for Types 2 to 5, 8, 9, 11 and 12, the factor β is the ratio of the actual end moment to the fixed end moment; and (3) for Types 6, 7 and 10, the factor a is the distance from a concentrated load to the nearest vertical support (see Table 1). The moment distribution types selected are believed to be common enough to correspond with typical design loading cases and cover a broad enough range to lead to general conclusions.

The original objective of this study was to examine the adequacy of the equivalent moment factor specified in Canadian standard CAN/CSA-S16-01. During the course of this examination, a critical evaluation of other published methods was also conducted. Not only are the shortcomings of the CAN/CSA-S16-01 procedure clarified, but a broad collection of solutions determined by other methods is also presented herein to illustrate their performance and limitations. Although relevant physical test data are scarce, numerical data are included as reference values where available. Finally, a new equivalent moment factor equation is proposed based on the findings of the investigation that incorporates the best features observed in the various existing methods. It is demonstrated that the proposed equation effectively rectifies current CAN/CSA-S16-01 deficiencies and produces accurate, yet conservative, approximations to the numerical solutions over a wide range of moment distribution types.

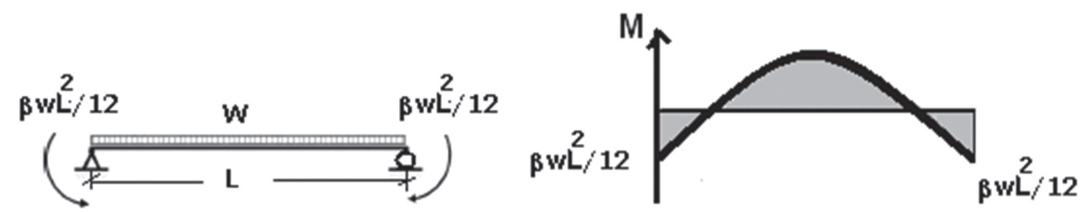
PROCEDURES PUBLISHED IN THE GENERAL LITERATURE

The fundamental aspects that characterize the non-uniform moment effect are the rate of change of the moment along the beam length, the number of curvature reversals between

Table 1. Types of Moment Distributions Considered in the Study*

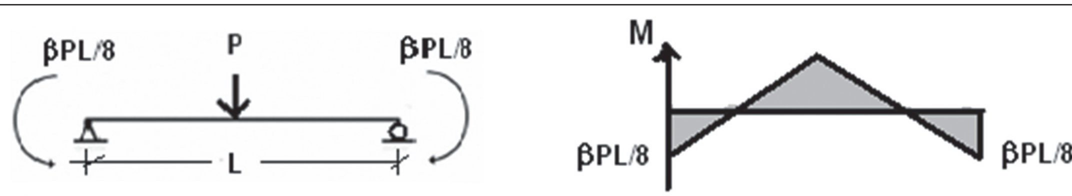
Type 1—End Moments Only

Type 2—Uniformly Distributed Load with Equal End Moments

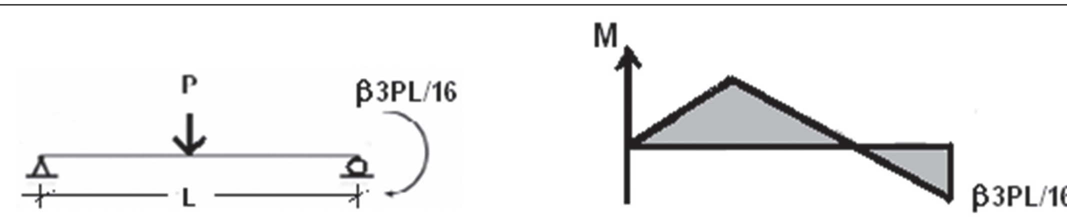


Type 3—Uniformly Distributed Load with One End Moment

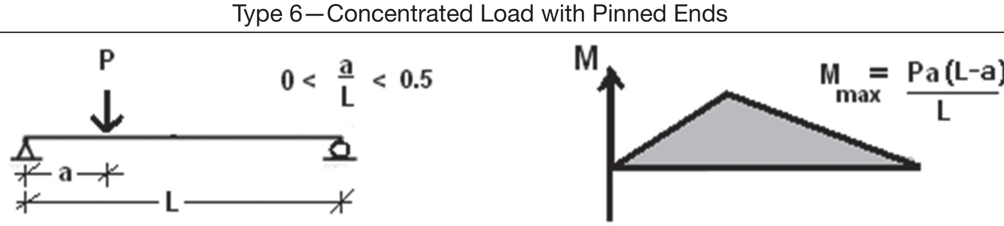
Type 4—Mid-span Concentrated Load with Equal End Moments



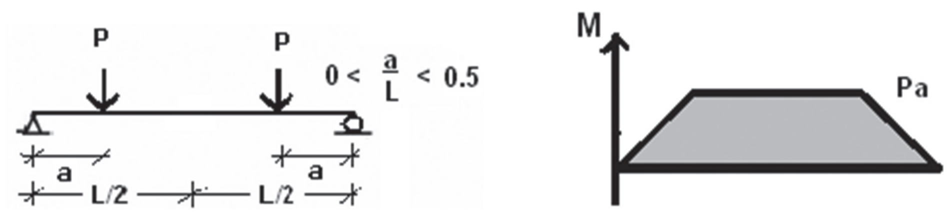
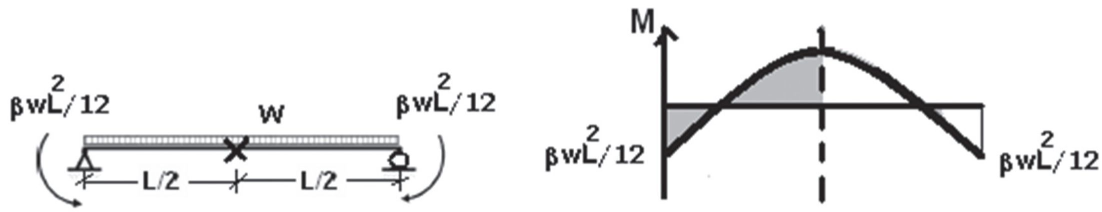
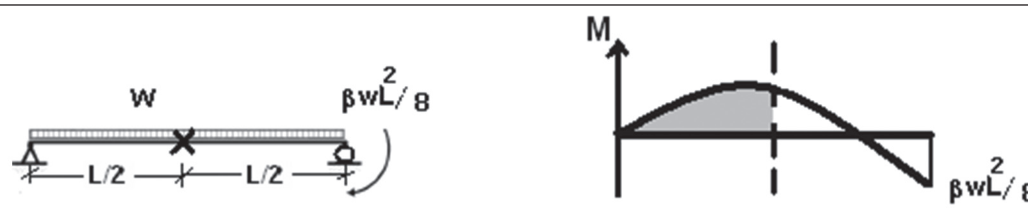
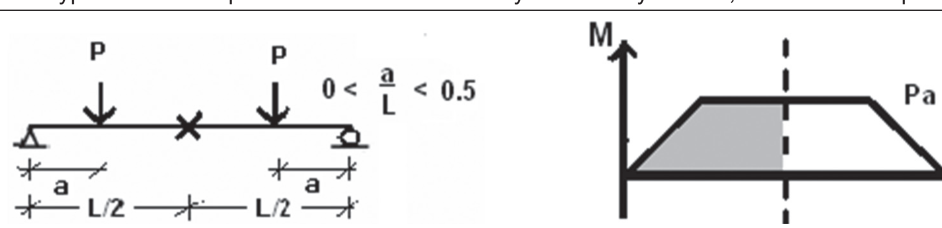
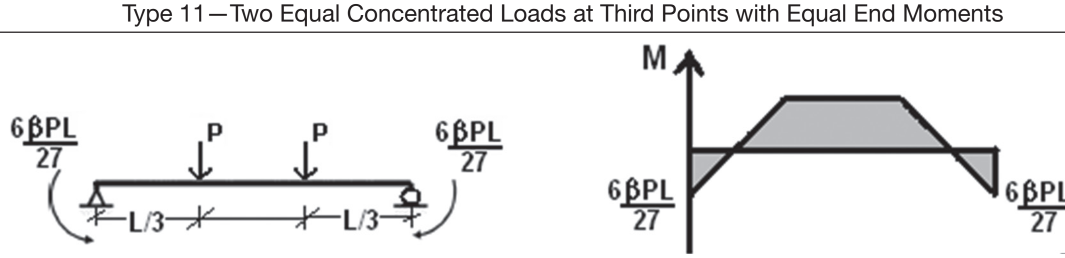
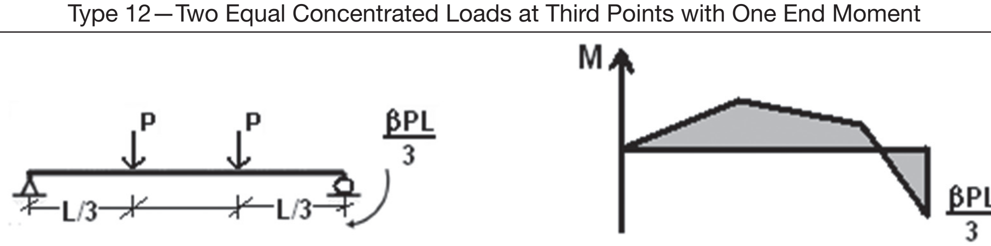
Type 5—Mid-span Concentrated Load with One End Moment



Type 6—Concentrated Load with Pinned Ends



* The ends of the members depicted are brace points, as well as points denoted by the symbol x.

Table 1 (cont.). Types of Moment Distributions Considered in the Study*
Type 7—Two Equal Concentrated Loads Symmetrically Placed with Pinned End

Type 8—Uniformly Distributed Load with Equal End Moments, Brace at Mid-span

Type 9—Uniformly Distributed Load with One End Moment, Brace at Mid-span

Type 10—Two Equal Concentrated Loads Symmetrically Placed, Brace at Mid-span

Type 11—Two Equal Concentrated Loads at Third Points with Equal End Moments

Type 12—Two Equal Concentrated Loads at Third Points with One End Moment

* The ends of the members depicted are brace points, as well as points denoted by the symbol x.

brace points, and the distance between the maximum moment and the brace locations. Due to the challenge of developing a simple procedure for determining equivalent moment factors that can capture all three aspects concurrently for all kinds of moment distributions, not all published methods are applicable to all moment distribution types. Available methods for determining equivalent moment factors can be divided into three categories:

1. Methods developed for unequal end moments only (e.g., Salvadori, 1955; Austin, 1961).
2. Methods developed for a general moment distribution (e.g., Kirby and Nethercot, 1979; Serna et al., 2006)
3. Methods developed to address numerous specific moment distributions individually (e.g., Trahair, 1993; Clark and Hill, 1960; Suryoatmono and Ho, 2002).

Methods Developed for Unequal End Moments Only

Two equations that are commonly used to approximate the effect of a constant moment gradient between brace points on the critical elastic moment are Equations 2 and 3 shown in Table 2. The parameter κ quantifies the influence of the flange force variation between the two ends. That is, if a beam flange is subjected to a bending-induced compression that varies between lateral supports, the degree of variation dictates the tendency of the beam to buckle elastically (Zuraski, 1992). Furthermore, if the flange force varies between tension and compression (i.e., when the unbraced segment of the beam is in double curvature), the beam is even less susceptible to lateral-torsional buckling. Equation 2 represents a lower bound to the original solutions developed by Salvadori (1955) using the Rayleigh-Ritz method, and Equation 3 is from the work of Austin (1961) for in-plane bending of beam-columns. Equation 3 is considered inappropriate for assessing out-of-plane buckling due to flexure alone because it is derived for members subjected to both axial load and bending simultaneously (AISC, 2005b), and, as such, it is not considered further.

Methods Developed for a General Moment Distribution—Quarter-Point Moment Methods

Equation 4, shown in Table 2, was developed to be applicable to all types of moment distributions (Kirby and Nethercot, 1979). It utilizes the magnitudes of the bending moments at four specific locations along the unbraced segment: the quarter point, M_a, centerline, M_b, third-quarter point, M_c, and maximum, M_{max}, moments. Equations with this format are referred to as the "quarter-point moment methods" in this paper. The main function of these four moments is to describe the degree of non-uniformity of the moment along the unbraced length, thus approximating its influence on the critical moment. Although not specified explicitly in the original publication, it has been indicated in numerous subsequent publications that using the absolute values of these moments in the equation is appropriate. Unlike Equations 2 and 3, the quarter-point moment methods are independent of the magnitudes of the end moments, unless one or both are also the maximum moment in the unbraced segment.

Another quarter-point moment method was developed by Serna et al. (2006) by curve fitting their numerical analysis results that account not only for the effect of a non-uniform moment distribution, but also the lateral, rotational and warping restraints at the brace points. Since the latter influences are not within the scope of this paper, the equation is written in a simplified form for laterally and torsionally simple end conditions as Equation 5 in Table 2. The main distinction of this equation as compared to Equation 4 is that the individual moment terms are squared and a square root format is assigned.

Methods Developed for Specific Moment Distributions

Clark and Hill (1960) and Nethercot and Trahair (1976) each published a list of equivalent moment factor values based on numerical analyses for specific non-uniform moment distributions, as shown in Table 3. Although not applicable to all typical design loading cases, they provide a good database from which designers can approximate equivalent moment factors for other distributions. The two sets are nearly identical, except for the value in the Type 2 distribution when β equals 1.0. For this case, the value of 1.30 from Clark and Hill (1960) appears to be incorrect, and if it is recalculated using the original source of data, a value of 2.52 is obtained, as reflected in Table 3.

Instead of discrete values, Trahair (1993) published individual equations (see Table 3) for several moment distribution types based on curve fitting of numerical data. These equations apply to a much wider range of moment distributions than do the lists of discrete values because the designer can adjust the point load location along a beam or the magnitude of the end moments. Nethercot and Rockey (1972) also proposed a C_b equation for moment Type 7 that is a function of the distance between the point load and the closer support. Analogous equations presented by Suryoatmono and Ho (2002) are relatively complex as compared to those of Trahair (1993), and they address moment Types 1 through 3 only; therefore, they are not included in the comparisons in this paper.

PROCEDURES IN DESIGN SPECIFICATIONS

Because the methodologies discussed in the previous section have various degrees of practicality, accuracy, consistency and computational complexity, different steel design specifications have adopted different procedures for determining the equivalent moment factor to be used for the design of laterally unsupported beams. Nevertheless, the majority use a single method that is intended to be applicable to all types of moment distributions.

Table 2. Equivalent Moment Factor Equations Evaluated in This Study		
Equation [a]	Publication	Equivalent Moment Factor Equation
2	Salvadori (1955) [b, c]	$C_b = 1.75 + 1.05\kappa + 0.3\kappa^2 \le 2.3$
3	Austin (1961)	$C_b = (0.6 - 0.4\kappa)^{-1} \le 2.5$
4	Kirby and Nethercot (1979)	$C_b = \dfrac{12M_{max}}{2M_{max} + 3M_a + 4M_b + 3M_c}$
5	Serna et al. (2006)	$C_b = \sqrt{\dfrac{35M_{max}^2}{M_{max}^2 + 9M_a^2 + 16M_b^2 + 9M_c^2}}$
6	AISC Specification	$C_b = \dfrac{12.5M_{max}}{2.5M_{max} + 3M_a + 4M_b + 3M_c} \le 3.0$
7	British Standard BS 5950-1	$C_b = \dfrac{M_{max}}{0.2M_{max} + 0.15M_a + 0.5M_b + 0.15M_c} \le 2.273$
8	Australian Standard AS4100	$C_b = \dfrac{1.7M_{max}}{\sqrt{M_a^2 + M_b^2 + M_c^2}} \le 2.5$
9	Proposed Equation	$C_b = \dfrac{4M_{max}}{\sqrt{M_{max}^2 + 4M_a^2 + 7M_b^2 + 4M_c^2}} \le 2.5$

[a] Equation 1, specified in the body text, is the general lateral-torsional buckling equation to which these C_b equations are applied.
[b] Adopted by CAN/CSA-S16-01 and CAN/CSA-S6-06, but with an upper limit of 2.5.
[c] Adopted by AASHTO with additional requirements.

CAN/CSA-S16-01—Limit States Design of Steel Structures (CSA, 2001)

This Canadian design standard specifies Equation 2 in Table 2 for determining the equivalent moment factor for unbraced beam segments subjected to end moments, except that an upper limit of 2.5 is used instead of 2.3. It is not clearly stated in the standard whether or not this is intended to apply to beams that are also subjected to transverse loads within this length. Although the sign of κ is assigned as described previously, no indication is given of how to account for triple curvature (e.g., Moment Type 2 in Table 1). The standard also specifies that for non-linear moment distributions where "the bending moment at any point within the unbraced length is larger than the larger end moment" (CSA, 2001), the equivalent moment factor be taken as 1.0. This additional clause effectively requires the designer to ignore the beneficial effect of a non-uniform moment distribution under this common circumstance. Moreover, if the standard is interpreted to mean that Equation 2 applies just for cases of end moments only (without any transverse loads), then it is silent on how to account for non-uniform moment distributions that are not captured by this additional clause.

CAN/CSA-S6-06—Canadian Highway Bridge Design Code (CSA, 2006)

This standard has adopted the same procedure as CAN/CSA-S16-01; however, the commentary to the standard refers users to the procedures of Clark and Hill (1960) as an alternative approach.

ANSI/AISC 360-05—Specification for Structural Steel Buildings (AISC, 2005a)

The American steel design specification for buildings stipulates that Equation 6 in Table 2 be used to determine the equivalent moment factor. As shown in the table, the only differences between Equations 4 and 6 are the coefficients for the terms M_{max}. This adjustment was made in an attempt

Table 3. Discrete Equivalent Moment Factors and Equations from the Literature [a]

Moment Type	Clark & Hill (1960)	Nethercot & Trahair (1976)	Trahair (1993)	Eurocode 3 (ECS, 1992)
1	$\kappa = -0.5$ $C_b = 1.31$ $\kappa = 0$ $C_b = 1.77$ $\kappa = 0.5$ $C_b = 2.33$ $\kappa = 1.0$ $C_b = 2.56$	$\kappa = 0$ $C_b = 1.75$ $\kappa = 1.0$ $C_b = 2.56$	Equation 2 (Table 2)	$\kappa = -0.75$ $C_b = 1.141$ $\kappa = -0.5$ $C_b = 1.323$ $\kappa = -0.25$ $C_b = 1.563$ $\kappa = 0$ $C_b = 1.879$ $\kappa = 0.25$ $C_b = 2.281$ $\kappa = 0.5$ $C_b = 2.7$ $\kappa = 0.75$ $C_b = 2.927$ $\kappa = 1.0$ $C_b = 2.752$
2 [b]	$\beta = 0$ $C_b = 1.13$ ~~$\beta = 1.0$ $C_b = 1.30$~~ $\beta = 1.0$ $C_b = 2.52$	$\beta = 0$ $C_b = 1.13$ $\beta = 1.0$ $C_b = 2.58$	Numerical result: $\beta = 0$ $C_b = 1.09$ Equations: For $0 \le \beta < 0.75$, $C_b = 1.13 + 0.12\beta$ For $0.75 \le \beta \le 1.0$, $C_b = -2.38 + 4.8\beta$	$\beta = 0$ $C_b = 1.13$ ~~$\beta = 1.0$ $C_b = 1.285$~~ $\beta = 1.0$ $C_b = 2.52$
3	same as Type 2 for $\beta = 0$	same as Type 2 for $\beta = 0$	Numerical result: $\beta = 0$ $C_b = 1.09$ Equations: For $0 \le \beta < 0.7$, $C_b = 1.13 + 0.1\beta$ For $0.7 \le \beta \le 1.0$, $C_b = -1.25 + 3.5\beta$	same as Type 2 for $\beta = 0$
4	$\beta = 0$ $C_b = 1.35$ $\beta = 1.0$ $C_b = 1.70$	$\beta = 0$ $C_b = 1.35$ $\beta = 1.0$ $C_b = 1.70$	Numerical result: $\beta = 0$ $C_b = 1.31$ Equation: For $0 \le \beta \le 1.0$, $C_b = 1.35 + 0.36\beta$	$\beta = 0$ $C_b = 1.365$ $\beta = 1.0$ $C_b = 1.565$
5	same as Type 4 for $\beta = 0$	same as Type 4 for $\beta = 0$	Numerical result: $\beta = 0$ $C_b = 1.31$ Equations: For $0 \le \beta < 0.89$, $C_b = 1.35 + 0.15\beta$ For $0.89 \le \beta \le 1.0$, $C_b = -1.2 + 3.0\beta$	same as Type 4 for $\beta = 0$
6	$a = L/2$ same as Type 4 for $\beta = 0$	$a = L/4$ $C_b = 1.44$ $a = L/2$ $C_b = 1.35$	Numerical result: $a = L/2$ $C_b = 1.31$ Equation: For $0 \le a/L \le 0.5$, $C_b = 1.35 + 0.4(1-2a/L)^2$	$a = L/2$ same as Type 4 for $\beta = 0$
7 [c]	$a = L/4$ $C_b = 1.04$ $a = L/2$ same as Type 4 for $\beta = 0$	$a = L/4$ $C_b = 1.04$ $a = L/2$ same as Type 4 for $\beta = 0$	Numerical result: $a = L/2$ $C_b = 1.31$ Equation: For $0 \le a/L \le 0.5$, $C_b = 1.0 + 0.35(2a/L)^2$	$a = L/4$ $C_b = 1.046$ $a = L/2$ same as Type 4 for $\beta = 0$

[a] No C_b values or equations were published for moment Types 8 to12.
[b] Strikethrough indicates error in original reference; refer to text for clarification.
[c] For moment Type 7 only, Nethercot and Rockey (1972) propose $C_b = 1.0 + (a/L)^2$.

to give better results for cases of fixed end supports (AISC, 2005b). For design purposes, this specification sets an upper limit to the equivalent moment factor of 3.0, which is the highest among all specifications discussed here. The commentary to the specification indicates that Equation 2 is also appropriate for cases where the moment distribution is linear between brace points.

AASHTO—LRFD Bridge Design Specifications (AASHTO, 2007)

Similar to the current CAN/CSA-S16-01 procedure, this specification uses Equation 2 as the primary equivalent moment factor equation and also specifies that the value be taken as 1.0 whenever the larger end moment is not the largest moment throughout the unbraced segment. However, the main difference between the two specifications is that in order to avoid the non-conservative results obtained when Equation 2 is used for certain non-uniform moment distributions, AASHTO (2007) introduces an equivalent linear moment distribution. The larger end moment and the mid-span moment are projected back linearly to obtain an imaginary smaller end moment, and then the larger value of the actual and imaginary smaller end moment is used to determine κ in Equation 2. Figure 1 illustrates an example for which a more appropriate solution is obtained if the magnitude of the mid-span moment is taken into consideration. The AASHTO (2007) procedure requires the calculation of two different equivalent moment factors—one for each of the top and bottom flanges—if both flanges experience compression due to reversing curvatures.

BS 5950-1—Structural Use of Steelwork in Building: Code of Practice for Design (BSI, 2000)

The British standard specifies Equation 7 in Table 2 for determining the equivalent moment factor. Among all specifications discussed in this study, BS 5950-1 has the lowest upper limit (2.273).

Eurocode 3 EN-1993-1-1—Design of Steel Structures (ECS, 1992)

In Annex F of the European design code, tabulated discrete equivalent moment factors are provided for moment Types 1, 2, 4 and 7 (see Table 1). These values are similar to those published by Clark and Hill (1960), as shown in Table 3. As in the original publication, the value for moment Type 2 when β equals 1.0 appears to be in error, and it has been corrected accordingly in Table 3.

AS 4100—Australian Standard: Steel Structures (SAA, 1998)

The Australian design standard specifies Equation 8 in Table 2 for determining the equivalent moment factor. Similar to Equation 5, it employs a square root format.

ASSESSMENT APPROACH

Equivalent moment factors for 12 diverse selections of bending moment distribution types have been determined from the various equations discussed in this paper. These solutions are compared against each other, as well as with numerical results presented in the literature. Although White and Kim (2008) included the results of hundreds of physical test results collected from numerous sources in a comprehensive statistical study on the flexural resistance of beams, the majority of these do not fall within the scope of the current study because the experiments either involved transverse loading applied above or below the shear center, were conducted on beams with mono-symmetric or hybrid cross-sections, were influenced by interaction buckling, or resulted in inelastic global or local buckling. Nonetheless, they included four experimental results from simply supported beams tested with a mid-span concentrated load applied through the shear center (i.e., moment Type 6, $a/L = 0.5$) that failed by elastic lateral-torsional buckling. Since there are so few suitable test results available, and because this particular moment distribution is associated with a relatively well-established equivalent moment factor, their inclusion would add little to the discussion presented herein. Therefore, these four tests are excluded from the comparisons in this paper.

Representative Moment Distributions

As shown in Table 1, the bending moment types considered in this study have been selected to cover a broad variety of potential situations. Moreover, each moment type envelopes a wide range of moment diagrams by varying either the magnitude of the end moments or the concentrated load locations.

The value of κ for Type 1 (linear) moment distributions reflects the ratio of the end moments and can therefore vary only from −1.0 to 1.0. The variable β, used for moment Types 2 to 5, 8, 9, 11 and 12, was assigned to alter the magnitude of the end moments. When β is set to 0, it represents a pinned (in plane) boundary condition, whereas when it is equal to 1.0, it represents a fixed boundary condition. The

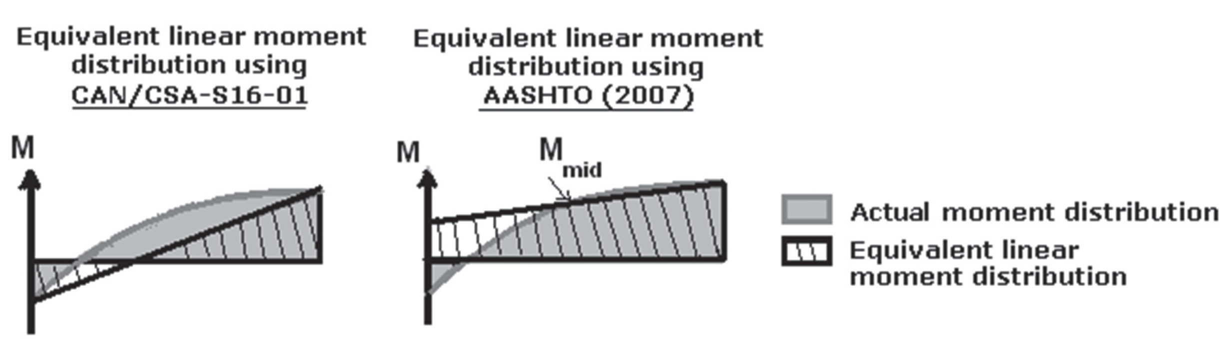


Fig. 1. Simplified moment diagrams according to CAN/CSA-S16-01 and AASHTO.

factor β was considered over a range from negative to a large enough positive value so that the scenarios of single, double and triple flexural curvature, combined with either a maximum moment at the end or away from the braces, were all covered and examined. Moment diagrams with negative values of β or values greater than 1.0 provide useful insight for evaluating continuous span structures. The variable a, used for moment Types 6, 7 and 10, was assigned to provide a means for altering the point load location along an unbraced beam segment. As the location of the concentrated load coincides with the maximum moment for these moment distribution types, this variation helps to develop trends of solutions that depict the influence of a varying distance between the point of maximum moment and the brace points.

Moment Types 2, 4 and 11 were selected, in part, because they include moment distributions that correspond to triple curvature. Due to the ambiguous instruction provided in standard CAN/CSA-S16-01 regarding the sign of κ for segments in triple curvature, solutions calculated by setting κ to both negative and positive are compared.

Moment Types 8, 9 and 10 are transformations of Types 2, 3 and 7, respectively, that simulate situations in which an extra brace is placed in the middle of the original unbraced segment. Moment Types 8 and 10 are included in the analysis to create circumstances where the moment is relatively uniform over much of the length, but no moment within the unbraced segment exceeds the larger end moment. As noted previously, this situation is not covered explicitly in CAN/CSA-S16-01. Unlike Types 8 and 10, the moment distributions of the left and right segments for moment Type 9 are different; thus, two sets of equivalent moment factor results are evaluated.

Numerical Simulation Data

Numerical analysis results from six different research programs are included in the comparisons as reference solutions. No numerical simulations were found in the literature for moment Types 8 to 12.

A total of 1500 critical bending stresses calculated using numerical analysis were tabulated by Austin et al., (1955). This extensive set of results was created by determining the critical moment of wide-flange sections with 10 different levels of flexural slenderness. Two loading cases—a uniformly distributed load and a mid-span point load—and 25 discrete levels of in-plane and out-of-plane end rotational restraint combinations were used. Moreover, three discrete levels of load application (i.e., load applied at the top flange, shear center, and bottom flange) were evaluated for each combination of the loading and boundary conditions. Because the effects of the height of load application and the out-of-plane rotational end restraint are not considered in the current study, only 10 out of the 1,500 solutions are used in the comparisons. Half of these solutions are selected from numerical models subjected to a uniformly distributed load (Type 2) and the other half are from models subjected to a mid-span point load (Type 4). The solutions selected are based on cross-sectional properties similar to the two distinct numerical models used in the analyses of Serna et al. (2006).

Suryoatmono and Ho (2002) published a suite of finite difference solutions for a 10-m-long (32.8 ft) doubly-symmetric wide-flange section with several different moment types: Type 1 with κ varying from −1.0 to 1.0; Types 2 and 3 with β varying from 0 to 2.0; and Type 6 with $a = L/2$ (same as Types 4 and 5 with $\beta = 0$). A total of 38 data points are used in the comparisons.

Serna et al. (2006) published an extensive set of equivalent moment factors based on numerical results for moment Types 1 to 5, with various end support conditions. Only data associated with no end lateral rotational restraint and no warping restraint are used in the evaluation. These researchers analyzed two models with different flexural slenderness values to ensure that the effect of the flexural slenderness on the equivalent moment factors was observed. Only the lower value of C_b from the two models is utilized for each loading condition in this study. As such, a total of 67 data points are used.

Other numerical results used in the comparisons were published by Clark and Hill (1960), Nethercot and Trahair (1976), and Trahair (1993). They are summarized in Table 3.

RESULTS AND IMPORTANT OBSERVATIONS

Comparisons Among Methods

The equivalent moment factor values determined by the methods discussed previously for all 12 moment distribution types are graphically presented alongside available numerical results in Figures 2 to 14. The purposes of these comparisons are to identify deficiencies and strengths of the various methods and to propose a method that optimizes the trade-off between computational effort and accuracy over a broad range of moment distribution types.

Due to the large quantity of data assembled, for clarity of the graphs in Figures 2 through 14 not all results from the various methods and equations could be included. Therefore, methods that are deemed not to provide any particular insight are sometimes omitted. To further alleviate difficulties in interpreting the graphs due to congestion of the data, all numerical results use filled symbols so as to distinguish them from the open and unfilled symbols used for design specifications and other published equations. Where the solutions for Equations 6 through 9 are controlled by the prescribed upper limit in the relevant design specification, the curves above the limit are shown dashed to reflect the accuracy of the equations in the event that the limit should be modified or eliminated.

Figure 2 demonstrates that all methods provide satisfactory approximations to the numerical results for Type 1 (linear) moment distributions for κ values up to about 0.5. As expected, results calculated using the CAN/CSA-S16-01 equation closely match the numerical results over the full range. Among all the quarter-point moment equations, the AISC equation gives the most conservative results within the region $0.5 < \kappa < 1.0$, with differences up to 18% compared to the numerical results.

For moment Type 2, the beam segment is under triple curvature bending when β is greater than 0. Since CAN/CSA-S16-01 does not specify whether the sign of κ in the C_b equation should be positive or negative for such a case, both positive and negative values were used to develop two different sets of solutions for comparison. Nonetheless, both sets fail to follow the trend of the numerical solutions. As illustrated in Figure 3, all other methods produce reasonable approximations to the numerical data, with several utilizing a maximum permissible value to prevent the use of very large values in design. One exception is that the value suggested by Eurocode 3 for the fixed end moment case ($\beta = 1.0$) is very low. Although it provides an excellent representation of general trends, the equation proposed by Serna et al. (2006) appears to be too aggressive for design purposes in the

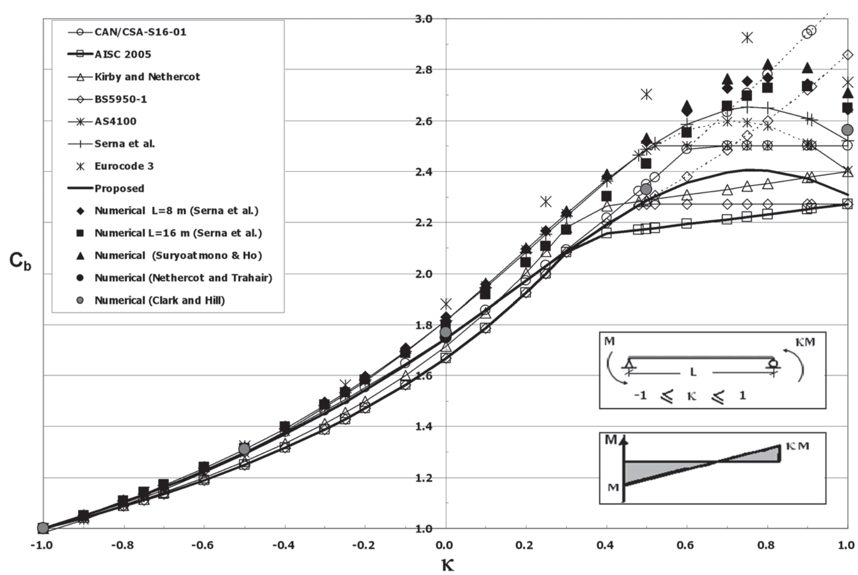


Fig. 2. C_b *Results for Moment Type 1.*

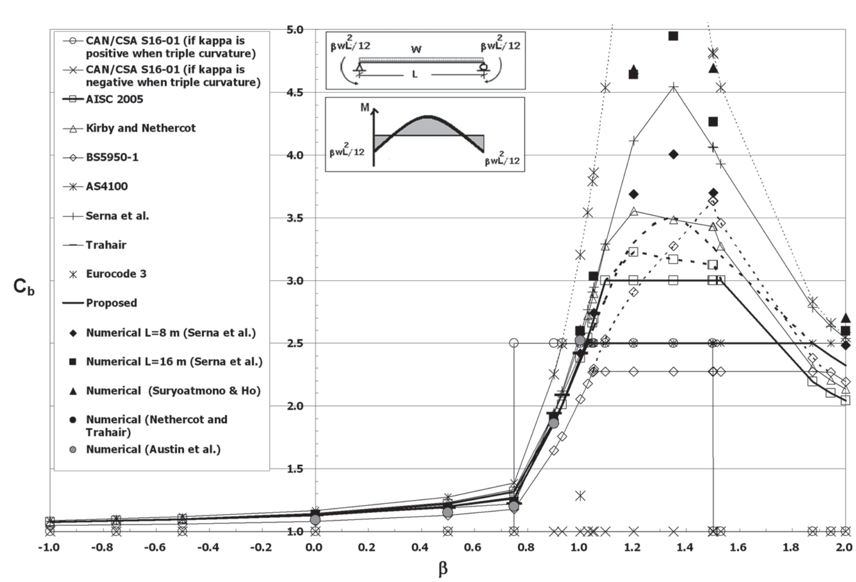


Fig. 3. C_b *Results for Moment Type 2.*

region 1.0 < β < 1.8 because the values obtained exceed many of their own numerical results.

The CAN/CSA-S16-01 results also fail to follow the trend of the numerical data for moment Type 3. Since there is no moment at the left end of the unbraced segment, the C_b equation in this standard always gives results equal to 1.75 unless the opposite end moment is not the maximum moment in the segment, in which case the value is 1.0. The abrupt transition between these two cases is at β = 0.69. As shown in Figure 4, non-conservative results exist where 0.69 < β < 0.85. Most other methods perform relatively well for this moment distribution type over the majority of the range of common β values. If the upper limits of the design equations are not considered, AS 4100 and Serna et al. (2006) seem to approximate the upper and lower bounds, respectively, of the numerical data in the upper range of β, while the AISC equation is the most conservative method in the same region.

As seen in Figure 5 for moment Type 4, results obtained from the Kirby and Nethercot (1979) and AISC formulae differ significantly from the numerical results on the non-conservative side in the region of 0.6 < β < 1.1 (restraint approaching a fixed end condition). Although the BS 5950-1

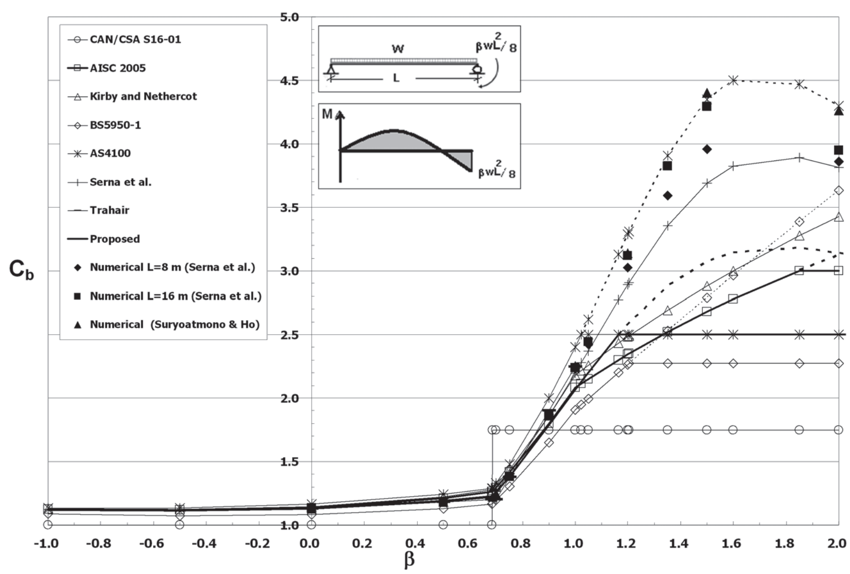


Fig. 4. C_b *Results for Moment Type 3.*

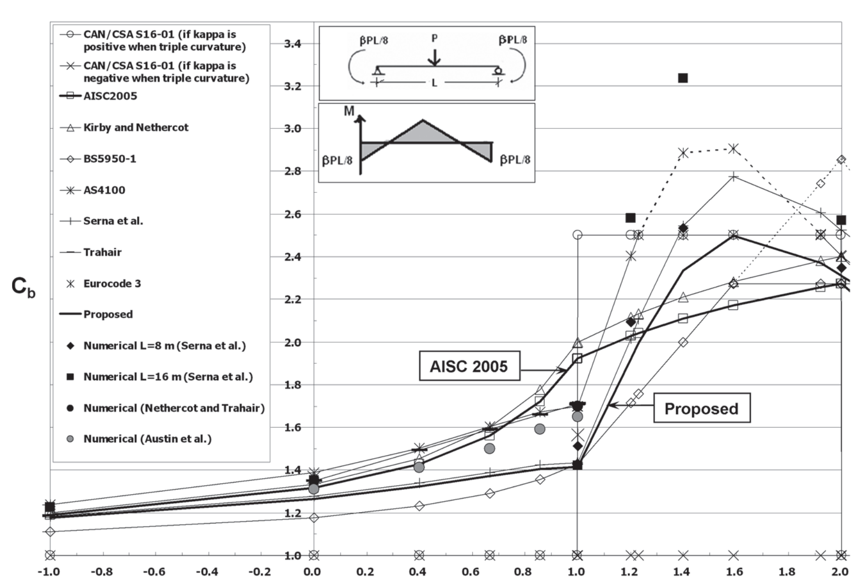


Fig. 5. C_b *Results for Moment Type 4*

equation is conservative in this region, it fails to capture the relatively abrupt change observed in the numerical data trends at about $\beta = 1.0$. Conversely, the AS 4100 and Serna et al. (2006) equations, which both use the square root format in the C_b equation, produce accurate approximations for this region and also capture the abrupt change in slope. The equation of Serna et al. (2006) is the more conservative of the two in this region. Similar to moment Type 2, two sets of solutions calculated using the CAN/CSA-S16-01 procedures are plotted for this moment type. Significantly non-conservative values are observed within the region $1.0 < \beta < 1.4$ if κ is taken positive for triple curvature. Otherwise, grossly conservative values are obtained throughout.

The C_b values obtained using CAN/CSA-S16-01 for moment Type 5 change abruptly from 1.0 to 1.75 when $\beta = 0.89$. Figure 6 shows that these results are dissimilar to all other methods. Other methods generally give reasonable results, with the BS5950-1 equation being the most conservative.

Since there are no end moments in moment Type 6, CAN/CSA-S16-01 sets C_b equal to 1.0, regardless of the location of the point load. Figure 7 shows that this solution is highly conservative in all situations. Although numerical results are

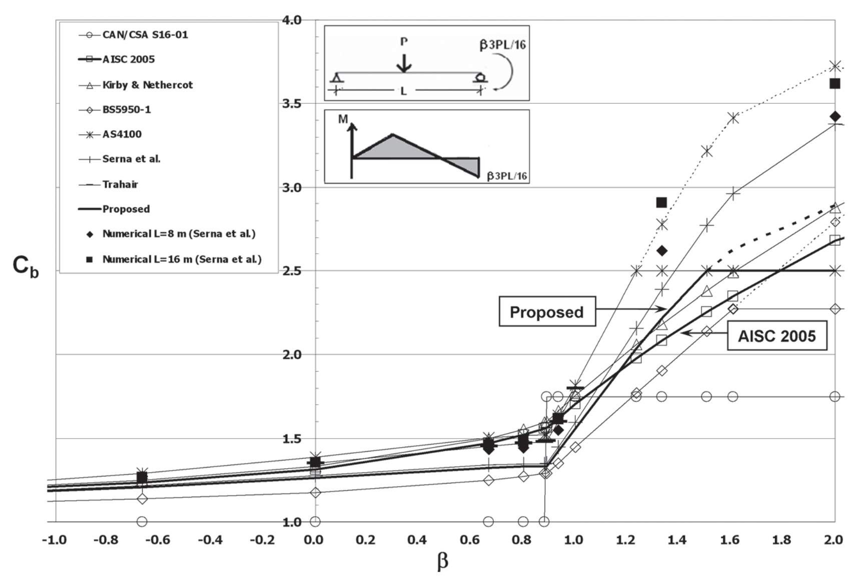


Fig. 6. C_b *Results for Moment Type 5.*

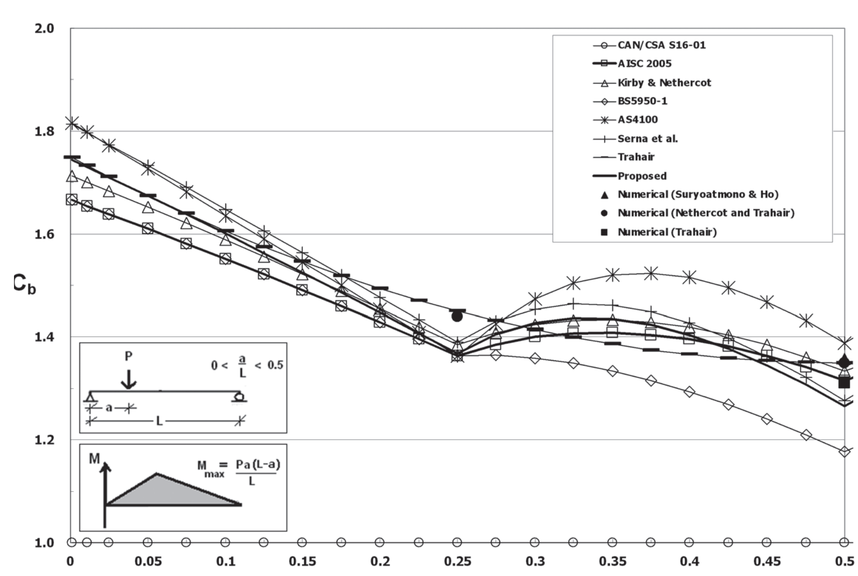


Fig. 7. C_b *Results for Moment Type 6.*

unavailable over the entire range of a/L, the actual trend of the solution can be reasonably predicted. The correct C_b solutions should decrease gradually from about 1.75 to 1.35 as the point load moves from one end of the unbraced segment toward mid-span. It is believed that the Trahair (1993) equation provides the closest approximation to the true solutions, although it is intended for use with this load case only. The local maxima in the curves predicted by most quarter-point moment methods at about $a/L = 0.35$ appear unreasonable, and in the case of the AS 4100 equation the C_b values around the peak are likely significantly non-conservative. Of all the quarter-point moment methods, BS5950-1 gives the most conservative results in this region.

Similar to moment Type 6, the CAN/CSA-S16-01 equation gives C_b values equal to 1.0 for the full range of a/L for moment Type 7 because no end moment is present. In this case, it is apparent that the correct solutions should increase gradually from 1.0 to about 1.35 as the point loads move from the ends of the unbraced segment toward mid-span, as obtained from the equations by Trahair (1993) and Nethercot and Rockey (1972) that were derived for this load case only. Figure 8 shows that all quarter-point moment methods

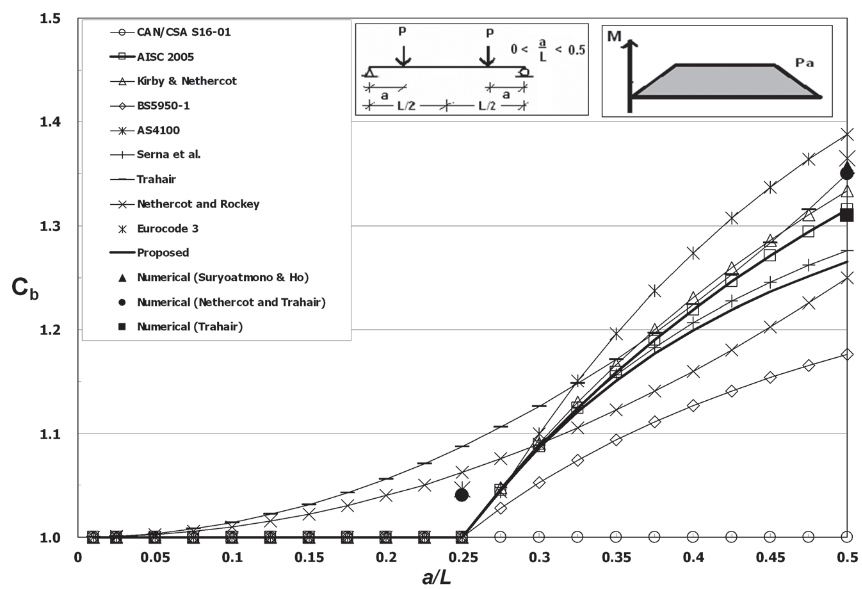


Fig. 8. C*b Results for Moment Type 7.*

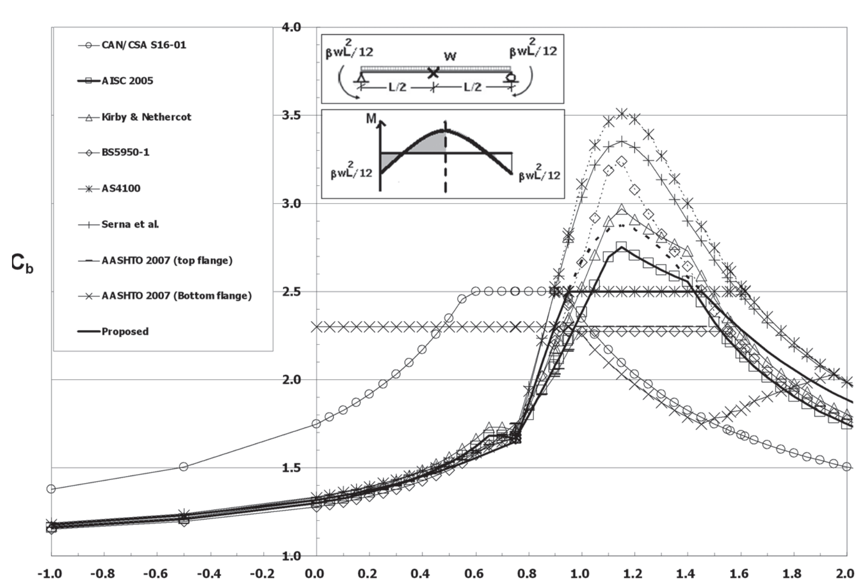


Fig. 9. C_b *Results for Moment Type 8.*

fail to reflect the effect of the non-uniform moment distribution when the point loads are between the end and the first quarter-point of the unbraced segment, although all methods appear to provide conservative solutions over the full range with the exception of AS 4100.

No numerical results are available for moment Types 8 to 12. However, evaluation of the performance of the various methods can be based on judgment and the knowledge obtained from the results observed for moment Types 1 to 7. Figure 9 (Type 8) and Figure 11 (Type 9, right unbraced segment) show that the C_b results obtained by CAN/CSA-S16-01 are much higher than the results of other methods for the ranges of $-1.0 < \beta < 0.85$ and $0 < \beta < 0.75$, respectively, and are considered to be highly non-conservative over most of these ranges. On the other hand, Figure 10 (Type 9, left unbraced segment) shows that it produces highly conservative results as compared to other methods. Solutions by other methods are, in general, consistent and appear to be reasonable approximations to the true solutions. The equation in BS5950-1 tends to be the most conservative of the quarter-point moment methods over much of the ranges, and especially when the maximum value is invoked.

As illustrated in Figure 12, the CAN/CSA-S16-01 equation gives a solution of 1.75 for moment Type 10, regardless

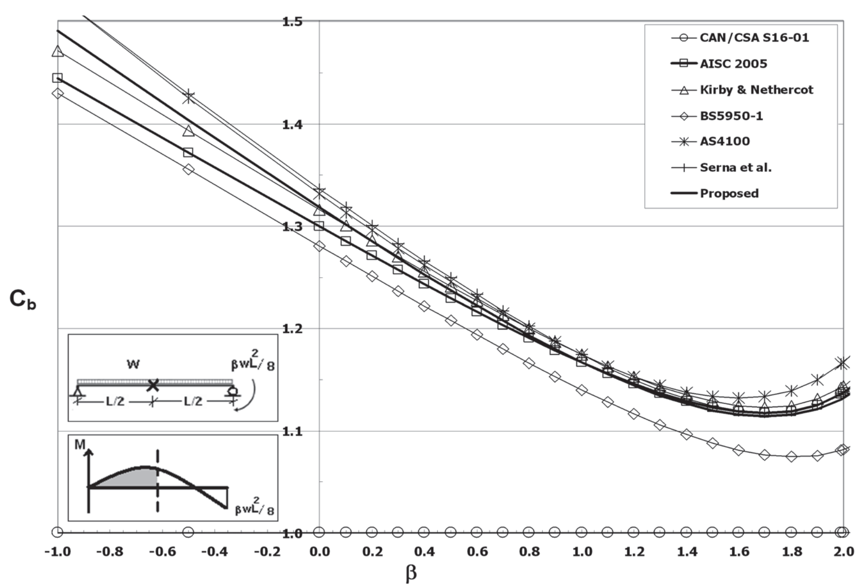


Fig. 10. C_b *Results for Moment Type 9, left unbraced segment.*

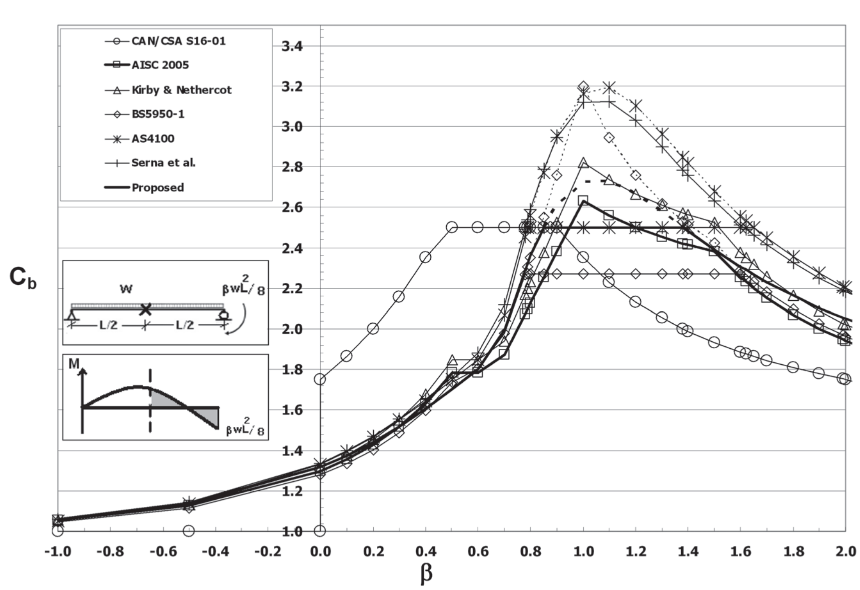


Fig. 11. C_b *Results for Moment Type 9, right unbraced segment.*

of the point load location. However, solutions determined by all other methods are much lower than 1.75 for the great majority of the range. It is evident in this case that the correct solutions should increase gradually from 1.0 to about 1.75 as the point loads move from the ends of the beam toward the braced mid-span. Although there are no numerical results to verify the correct solutions directly, it is clear that the solutions obtained using CAN/CSA-S16-01 are highly non-conservative. Conversely, the solutions of the AASHTO procedure, which uses the same equation as CAN/CSA-S16-01 but is based on an imaginary smaller end moment determined using the moment at the center of the unbraced segment, as described previously, are in better agreement with other methods. Again, BS5950-1 provides the most conservative solutions among the quarter-point moment methods, although all such methods provide similar results over the full range of *a/L*.

Solutions developed for moment Types 11 and 12 are illustrated in Figure 13 and Figure 14, respectively. Findings and observations are similar to those discussed previously for moment Types 2 and 3.

Although a broad investigation is presented herein that includes many procedures from the literature and design specifications, it is instructive to clarify the deficiencies of the

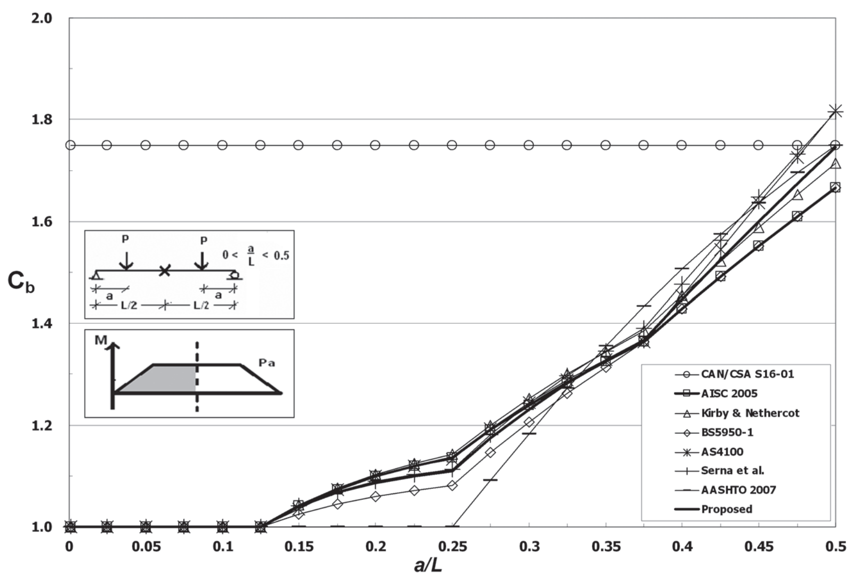


Fig. 12. C_b *Results for Moment Type 10.*

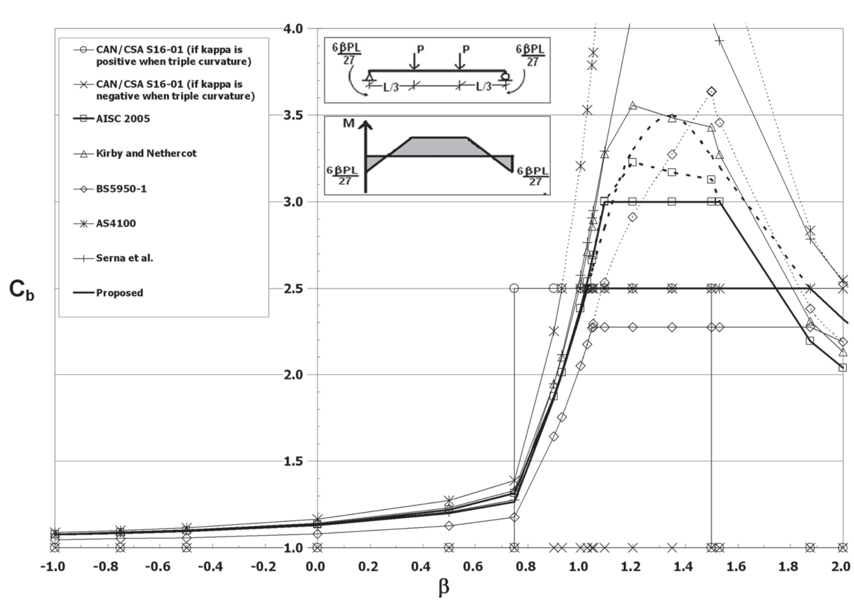


Fig. 13. C_b *Results for Moment Type 11.*

CAN/CSA-S16-01 method for determining equivalent moment factors. This design standard gives inconsistent results for all moment types evaluated in this study except for the linear moment distribution (Type 1), for which the procedure was originally derived. Driver and Wong (2007) summarize the ranges where this method produces acceptable results for each moment type presented here and conclude that the procedure is unsatisfactory over the full range in two out of the 12 types (Types 6 and 10) and over a significant part of the range in nine others. In all, they identified four general types of deficiencies in the CAN/CSA-S16-01 approach. First, its provisions tend to produce highly conservative results for simply supported beams that are unbraced between their ends because C_b always defaults to 1.0. Figure 7 indicates that C_b can be underestimated by more than 40% for moment Type 6, for example. Significantly conservative results can also occur in transversely loaded unbraced segments that experience either zero moment at one end of the segment (C_b = 1.75) or equal and opposite end moments (C_b = 1.0). Second, this method potentially overestimates C_b when a transversely loaded segment experiences a maximum moment at either end. For moment Type 10 (Figure 12), for example, the overestimation can be as high as 75%. Third, in 10 out of the 12 moment types discussed in this paper, C_b either remains unchanged over the entire range of *a/L* or β, or it experiences abrupt changes at particular β values, whereas for a gradually transforming moment distribution a gradually changing C_b function would appear more appropriate. This suggests that the CAN/CSA-S16-01 provisions, although not always producing non-conservative results, inconsistently accounts for the non-uniform moment distribution effect. Finally, CAN/CSA-S16-01 is ambiguous in some common design circumstances because it does not clearly state whether or not its provisions are applicable to an unbraced segment that is subjected to end moments in combination with other loading, or whether the sign of κ should be positive or negative for the case of triple curvature. The latter ambiguity in some cases creates a situation where the choice of sign results in either a highly conservative or a highly non-conservative solution. Driver and Wong (2007) provide a more detailed discussion of the CAN/CSA-S16-01 provisions.

Important Observations Concerning Quarter-Point Moment Methods

As shown in all 12 comparisons, the quarter-point moment methods, which are purported to be applicable for any moment distribution, tend to give reasonable results for different moment types even though their levels of accuracy and conservatism vary. The coefficients for the four discrete moments used in these equations are selected deliberately to weight the influence of each quarter-point moment magnitude relative to the maximum moment, and the coefficients selected are highly influential to the accuracy of the results. A few common characteristics of these coefficients are observed by examining Equations 4 to 8. For example, the sum of all coefficients in the denominator is always equal to the coefficient in the numerator. This condition ensures that C_b = 1.0 for a uniform moment distribution (i.e., when $M_a = M_b = M_c = M_{max}$). Also, the coefficient of M_a is identical to the coefficient of M_c to ensure that the C_b value is the same

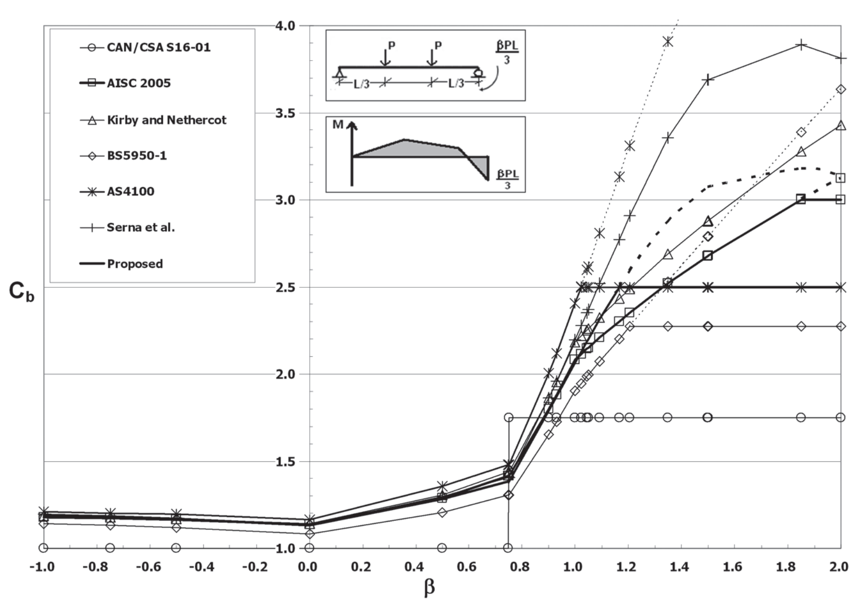


Fig. 14. C_b *Results for Moment Type 12.*

for any two mirrored moment distributions about the centerline of the unbraced segment. The last common characteristic is that the coefficient of M_b is always at least equal to that of M_a and M_c. This represents the fact that beams with moment distributions where the point of maximum moment is close to the centerline of the unbraced segment (i.e., $M_b \approx M_{max}$) are more prone to lateral-torsional buckling than those where it is close to the quarter points.

All quarter-point moment methods may fail to provide conservative approximations of the actual equivalent moment factor under the presence of abrupt changes in the moment diagram, i.e., for segments loaded with concentrated moments. Arguably, this condition is rare in typical design problems, but it can occur, for example, when a vertical cantilever post affixed to the beam flange is loaded parallel to the beam axis. Figure 15 demonstrates one situation where the accuracy of the quarter-point moment equations is questionable because they fail to capture the uniformity of the moment distribution between the quarter points. Using any of Equations 4 through 8 for the two different moment distributions shown in this example results in the same C_b value, although one case is clearly more critical than the other. If a designer were simply to set M_a and M_c equal to M_{max} instead of using the actual quarter-point values, Equations 4 to 8 produce the result $C_b = 1.0$, which may be highly conservative, depending on the actual locations of the concentrated moments. One way to address this deficiency would be to increase the number of moment parameters in the equation to better represent the actual moment distribution, but it would also increase the complexity of the equation as well as the concomitant computational effort required for design. Due to the relative rarity of these cases, this increase in complexity is likely unwarranted if designers are simply aware of cases where the quarter-point moment equations should be applied with due caution.

Another concern with the quarter-point moment equations arises because the resulting equivalent moment factor is independent of the sign of the internal moments. It is unclear how these equations can account for the effect of an abrupt reversal of curvature in a beam, such as the one illustrated in Figure 16, Case 2. Although it is apparent that Case 2 loading should result in a more favorable equivalent moment factor than Case 1 due to the presence of double curvature, all quarter-point moment equations incorrectly give the same C_b value for both diagrams because the absolute values of the moment parameters fail to distinguish between the two

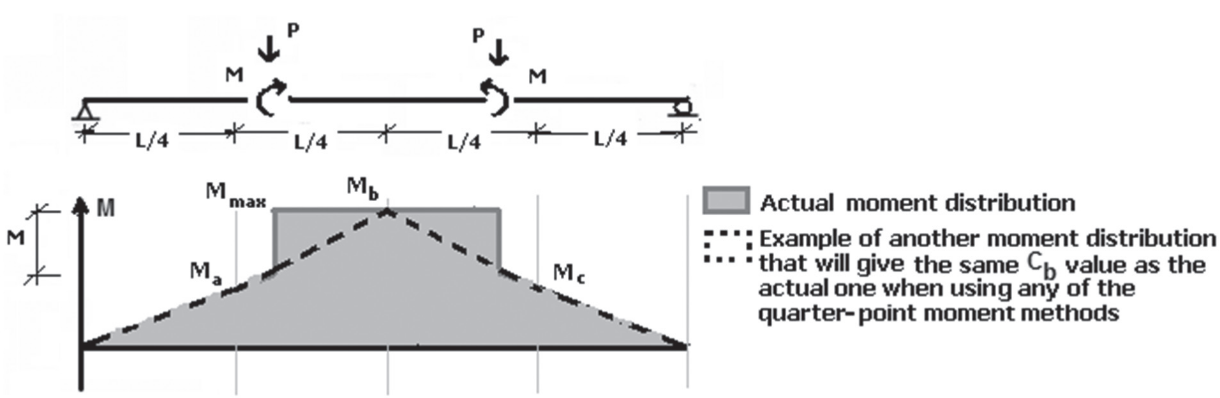


Fig. 15. Inaccuracy of quarter-point moment methods for case of abrupt change in moment.

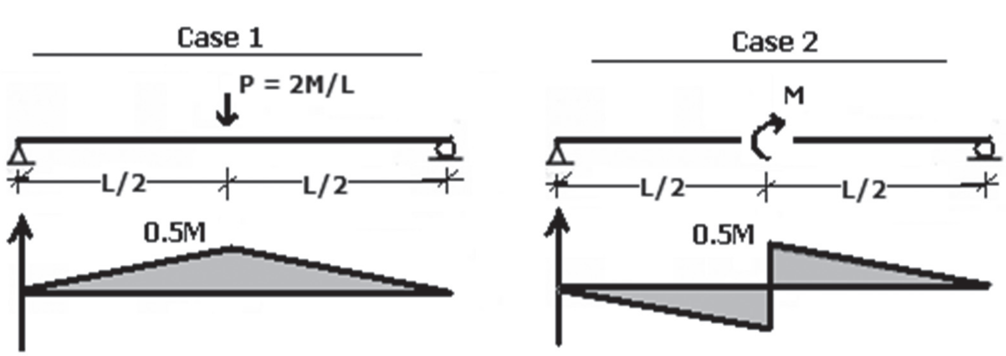


Fig. 16. Inaccuracy of quarter-point moment methods for case of abrupt curvature reversal.

cases. As a result, the solution for Case 2 is highly conservative. Again, refining the equations to rectify this shortcoming is likely unnecessary because this scenario is also relatively uncommon, but designers need to be aware of the limitations of the procedure.

PROPOSED EQUATION

It is evident from the foregoing discussion that modifications are required to improve the accuracy of the CAN/CSA-S16-01 equivalent moment factor procedures. All other methods considered in this investigation are more versatile; however, these methods have their own drawbacks. The method proposed by Trahair (1993) tends to provide very good results, but it relies on several independent equations, each having a limited scope of application, and therefore becomes somewhat cumbersome for general design purposes. The use of a table of individual C_b values for specific cases, similar to those of Clark and Hill (1960), Nethercot and Trahair (1976) and Eurocode 3, is considered undesirable for design specifications due to the innumerable common cases for which no guidance would be provided. Although the AASHTO procedure effectively eliminates many of the non-conservative results obtained from Equation 2 by using an equivalent linear moment diagram, it still gives highly conservative results for simply supported beams braced only at the ends and subjected to transverse loading. Despite their shortcomings for certain rare cases, as discussed in the previous section, the quarter-point moment approach shows the most promise of wide applicability, combined with simplicity, and for the most part these equations capture the trends observed in the numerical data well in the cases considered. Their accuracy, however, depends largely on the coefficients of the moment terms. The equation in the British standard tends to give very conservative results for several moment types. The Kirby and Nethercot (1979) and AISC equations are nearly equivalent and generally give good results. However, they are unable to capture the trends of the numerical data for the common case of moment Type 4 and give non-conservative results (up to about 32%) in the region of $0.6 < \beta < 1.1$.

Equations 5 and 8 (Table 2), by Serna et al. (2006) and specified in the Australian design standard, respectively, generally capture the C_b trends very well. Their ability to provide better results for moment Type 4 than the Kirby and Nethercot (1979) and AISC equations is attributed to the square root format that makes Equations 5 and 8 unique among the quarter-point moment equations considered in this investigation. However, both Equations 5 and 8 produce C_b values in some situations that exceed the numerical data significantly, thereby producing non-conservative results. Equation 8 gives results that exceed the numerical data for moment Types 2, 4 and 6, and in cases where no numerical data are available, it often produces the highest C_b values of all methods considered. Similar drawbacks exist for Equation 5, although many of the cases where the numerical data are exceeded are for the larger values of C_b that could be eliminated by using an upper bound on the permissible values. In light of the rather sparse set of corroborating numerical and experimental data available, both equations are judged to be too aggressive for design use. Therefore, a modified quarter-point moment equation utilizing the superior square root format is proposed for design in order to provide accurate C_b values and properly represent the data trends, while at the same time minimizing the chance of obtaining non-conservative beam capacities. This equation takes the following form:

$$C_b = \frac{4M_{max}}{\sqrt{M_{max}^2 + 4M_a^2 + 7M_b^2 + 4M_c^2}} \leq 2.5 \tag{9}$$

where the moment parameters are defined in the same way as for the other quarter-point moment equations discussed herein. The upper limit of 2.5 is selected to prohibit the use of very high C_b values in design, although a different upper limit could be selected and justified based on reliability considerations. Nevertheless, it must be noted that the proposed equation appears to produce good results even without this limit. Therefore, if the limit were to be increased or removed, the better performance of the proposed equation, as compared to the other quarter-point moment methods that use the square root format, becomes even more important.

As shown in Figures 2 through 14, the proposed equivalent moment factor equation (Equation 9) provides far better approximations to the numerical data (and to estimated correct solutions where no such data exist) than does the equation in CAN/CSA-S16-01, and it also addresses shortcomings of the other methods, while producing appropriately conservative C_b values for design. For instance, it effectively avoids producing the non-conservative results obtained by the AISC equation for moment Type 4 in the range $0.6 < \beta < 1.1$ (the AISC and proposed equation curves are plotted with a heavier line weight to facilitate comparison). The proposed equation also gives very good results for loading that produces a linear moment distribution between brace points (Type 1), as shown in Figure 2.

SUMMARY AND CONCLUSIONS

Numerous published methods and equations for determining equivalent moment factors used in evaluating the elastic critical moment of laterally unsupported beams have been compared for a wide variety of moment distribution types. The investigation revealed that the procedure used currently in the Canadian design standard produces unacceptable results for the majority of the common bending moment distributions considered. Not only does this method give grossly conservative results for many common cases, it also frequently gives unconservative results. Large abrupt changes in C_b values with only slight changes in the shape of the moment diagram were observed in 6 out of the 12 moment distribution comparisons, which contributes to the overall poor performance of the procedure. Moreover, it does not give clear direction as to the sign of κ when the beam is under triple curvature.

The study also revealed drawbacks inherent in other methods. Overall, the quarter-point moment equations developed for general moment distributions capture the trends of the numerical data reasonably well. However, the evaluations show that the Kirby and Nethercot (1979) and AISC (2005) equations produce non-conservative results in some situations, while the BS 5950-1 (BSI, 2000) equation, although generally conservative, produces comparatively less accurate results. The Serna et al. (2006) and AS 4100 (SAA, 1998) equations capture the trends of the numerical data more consistently by implementing a square root format in the quarter-point moment method. However, they produce results that exceed the numerical data in several cases, implying that both equations are too aggressive for design purposes.

To capture the best features of the various methods investigated, yet improve the overall suitability for general design purposes, a modified quarter-point moment equation using the square root format (Equation 9) is proposed. Not only does it simulate the trends of the numerical solutions closely, but it also produces reasonable and conservative equivalent moment factors, even in cases where other methods do not. Moreover, it is simple and well-suited to design applications. Like all quarter-point moment methods, the proposed equation does not produce good results in some situations where concentrated moments are applied. Nevertheless, it is believed to be appropriate for the vast majority of typical design cases.

REFERENCES

AASHTO (2007), *LRFD Bridge Design Specifications,* 4th edition, American Association of State Highway and Transportation Officials, Washington, DC.

AISC (2005a), *Specification for Structural Steel Buildings*, ANSI/AISC 360-05, American Institution of Steel Construction, Chicago, IL.

AISC (2005b), Commentary on the *Specification for Structural Steel Buildings*, American Institution of Steel Construction, Chicago, IL.

Austin, W.J. (1961), “Strength and Design of Metal Beam-Columns,” *Journal of the Structural Division*, American Society of Civil Engineers, Vol. 87, No. ST4, April, pp. 1–32.

Austin, W.J., Yegian, S. and Tung, T.P. (1955), “Lateral Buckling of Elastically End-Restrained I-Beams,” *Proceedings of the American Society of Civil Engineers*, Vol. 81, pp. 673-1–673-25.

BSI (2000), *Structural Use of Steelwork in Building: Code of Practice for Design, Rolled and Welded Sections,* BS5950-1, British Standards Institution, London, United Kingdom.

Clark, J.W. and Hill, H.N. (1960), “Lateral Buckling of Beams,” *Journal of the Structural Division,* American Society of Civil Engineers, Vol. 86, No. ST7, pp. 175–196.

CSA (2001), *Limit States Design of Steel Structures,* CAN/CSA-S16-01, Canadian Standards Association, Mississauga, ON.

CSA (2006), *Canadian Highway Bridge Design Code,* CAN/CSA-S6-06, Canadian Standards Association, Mississauga, ON.

Driver, R.G. and Wong, E. (2007), “Critical Evaluation of the CSA-S16-01 Equivalent Moment Factor for Laterally Unsupported Beams,” *2007 Annual General Meeting and Conference,* Canadian Society for Civil Engineering, Paper GC-189.

ECS (1992), *Eurocode 3: Design of Steel Structures, General Rules and Rules for Building,* EN-1993-1-1, European Committee for Standardization, Brussels, Belgium.

Galambos, T.V., Ed. (1998), *Guide to Stability Design Criteria for Metal Structures*, Chapter 12 “Bracing,” Structural Stability Research Council, John Wiley and Sons, Inc., New York, NY.

Kirby, P.A. and Nethercot, D.A. (1979), *Design for Structural Stability,* Halsted Press, New York, NY.

Nethercot, D.A. and Rockey, K.C. (1972), “A Unified Approach to the Elastic Lateral Buckling of Beams,” *Engineering Journal*, American Institute of Steel Construction, July, pp. 96–107.

Nethercot, D.A. and Trahair, N.S. (1976), “Lateral Buckling Approximations for Elastic Beams,” *The Structural Engineer*, Vol. 54, No. 6, pp. 197–204.

SAA (1998), *Australian Standard—Steel Structures*, AS4100, Standards Australia, Homebush, NSW, Australia.

Salvadori, M.G. (1955), “Lateral Buckling of I-Beams,” *ASCE Transactions*, American Society of Civil Engineers, Vol. 120, pp. 1165–1177.

Serna, M.A., López, A., Puente, I. and Yong, D.J. (2006), “Equivalent Uniform Moment Factors for Lateral-Torsional Buckling of Steel Members,” *Journal of Constructional Steel Research*, Vol. 62, pp. 566–580.

Suryoatmono, B. and Ho, D. (2002), "The Moment–Gradient Factor in Lateral–Torsional Buckling on Wide Flange Steel Sections," *Journal of Constructional Steel Research*, Vol. 58, pp. 1247–1264.

Trahair, N.S. (1993), *Flexural–Torsional Buckling of Structures*, CRC Press, Boca Raton, FL.

White, D.W. and Kim, Y.D. (2008), "Unified Flexural Resistance Equations for Stability Design of Steel I-section Members: Moment Gradient Tests," *Journal of Structural Engineering*, American Society of Civil Engineers, Vol. 134, No. 9, pp. 1471–1486.

Zuraski, P.D. (1992), "The Significance and Application of C_b in Beam Design," *Engineering Journal*, American Institute of Steel Construction, Vol. 39, No. 1, pp. 20–25.